JOURNAL OF APPROXIMATION THEORY 36, 141–155 (1982)

The Set of Continuous Selections of a Metric Projection in $C(X)$

JÖRG BLATTER AND LARRY SCHUMAKER*

Instituto de Matemática e Estatística, Universidade de São Paulo, C.P. 20570 (Ag. Iguatemi), 01451 São Paulo, SP, Brasil, and Department of Mathematics and Center for Approximation Theory, Texas A & M University, College Station, Texas 77843, U.S.A.

Communicated by E. W. Cheney

Received October 25, 1981

1. INTRODUCTION

Throughout this paper we deal with approximation in the uniform norm of elements of the space $C(X)$ of all continuous real-valued functions on a compact hausdorff topological space X by elements of a finite-dimensional subspace G of $C(X)$. For $f \in C(X)$ we call

$$d(f) = \inf\{\|f-g\|: g \in G\}$$

the *distance of f from G* and

$$P(f) = \{ g \in G: \|f-g\| = d(f)\}$$

the *set of best approximations of f in G*. The set-valued mapping P which maps an $f \in C(X)$ onto the non-empty compact convex subset $P(f)$ of $C(X)$ is called the *metric projection of* $C(X)$ *onto G*.

In recent years, there has been considerable interest in continuous mappings $S: C(X) \to C(X)$ with the property that $Sf \in P(f)$ for every $f \in C(X)$. Such an S is called a *continuous selection for the metric projection P*.

To date, the available results on continuous selections for metric projections in $C(X)$ deal primarily with their existence. In particular, Lazar *et al.* [6] identified, for arbitrary X, the 1-dimensional subspaces G of $C(X)$ for which P admits a continuous selection, and, in the case that X is an interval, Nürnberger and Sommer in a series of papers (see [9] and the

* Supported in part by a grant from Fundação de Amparo à Pesquisa do Estado de São Paulo.

0021-9045/82/100141–15$02.00/0

references therein) characterized all (finite-dimensional) subspaces G of $C(X)$ for which P admits a continuous selection.

The purpose of this paper is to address the all but neglected question of the *uniqueness* of continuous selections. The main result of the paper is contained in Section 2, where, given that P admits at least one continuous selection, we compute for every $f \in C(X)$ the set

$$\bigcup \{Sf: S \text{ a continuous selection for } P\}. \tag{1}$$

This reduces the uniqueness question to the question of whether or not these sets are singletons for every $f \in C(X)$.

As applications of our result, in Sections 3 and 4 we settle the uniqueness question for arbitrary X if G is 1-dimensional and for arbitrary (finite-dimensional) G if X is an interval, respectively. We conclude the paper with several remarks.

2. The Main Result

Our approach to constructing the sets (1) will be based on two facts. The first is the following consequence of well-known results of Michael [7]:

> Suppose P admits a continuous selection and define a mapping Q by $Q(f) = \bigcup\{Sf: S$ a continuous selection for $P\}$ for $f \in C(X)$. Then Q is a lower semi-continuous mapping of $C(X)$ into the set of non-empty closed convex subsets of $C(X)$ and $Q(f) \subset P(f)$ for every $f \in C(X)$. Moreover, Q is the largest such mapping in the sense that every lower semi-continuous mapping R of $C(X)$ into the set of non-empty closed convex subsets of $C(X)$ which has the property that $R(f) \subset P(f)$ for every $f \in C(X)$ also has the property that $R(f \subset Q(f)$ for every $f \in C(X)$. (2)

In order to state the second fact, we require some additional notation. Given a function $f \in C(X)$ and a non-empty convex subset H of $P(f)$, we define the *set of (common) extreme points of* $f - H$ to be the set

$$E(f-H) = \{x \in X: |f(x) - g(x)| = d(f) \text{ for all } g \in H\}.$$

It is well known (see, e.g., [2]) that these sets are always non-empty and that all elements of H coincide on $E(f-H)$. It follows that if $f \notin G$, the set $E(f-H)$ is the disjoint union of the sets

$$E^+(f-H) = \{x \in X: f(x) - g(x) = d(f) \text{ for all } g \in H\}$$

and

$$E^-(f-H) = \{x \in X: f(x) - g(x) = -d(f) \text{ for all } g \in H\}.$$

The second fact we want to state is the following unpublished theorem of Blatter (see [1]):

> For $f \in C(X)$, P is lower semi-continuous at f iff there exists a neighborhood of $E(f-P(f))$ in which all elements of $P(f)$ coincide. (3)

With these two facts at hand, we can present the main idea of this paper: (A) by (2), the sets (1) can be alternatively described as maximal nonempty closed convex subsets of the sets $P(f)$ subject to lower semi-continuous dependence on f; (B) the construction of sets meeting this description is suggested by the pointwise criterion for lower semi-continuous dependence on f of the sets $P(f)$ themselves which is given in (3). We proceed to carry out this idea.

DEFINITION. Suppose S^* is a continuous selection for P. Then we define the *lower semi-continuous kernel* P^* *of* P *induced by* S^* as follows: Fix $f \in C(X)$. Set $H_0 = P(f)$ and, for $k = 1, 2, \ldots$,

$$H_k = \{ g \in H_{k-1}: g \text{ coincides with } S^*f \text{ in some neighborhood of } E(f - H_{k-1})\}.$$

It is easily verified that

> the sets H_k are all convex and
> $$P(f) = H_0 \supset H_1 \supset H_2 \supset \cdots \supset \{S^*f\}, \quad (4)$$
> for each $k = 1, 2, \ldots$, there exists a neighborhood of $E(f - H_{k-1})$ in which all elements of H_k coincide with S^*f, (5)
>
> for each $k = 1, 2, \ldots$, if H_k is a proper subset of H_{k-1}, then $\dim(H_k) < \dim(H_{k-1})$. (6)

It then follows that the sequence H_0, H_1, $H_2, \ldots$ is stationary from some point on. Let $k \geqslant 1$ be the smallest integer for which $H_k = H_{k-1}$, and set $P^*(f) = H_{k-1}$. It is clear from the properties of the sets H_k that $P^*(f)$ is a closed convex subset of $P(f)$ which contains S^*f, and that there exists a neighborhood of $E(f - P^*(f))$ in which all elements of $P^*(f)$ coincide with S^*f.

We now establish three lemmas, the last of which will be the key for our proof of the main result of this paper; the first two are modifications of lemmas in [2] and serve here only to prove the last.

LEMMA 1. *Let H be a subset of G, let E be a non-empty subset of X and let $g_0, h_0 \in H$ have the property that for every neighborhood U of E*

$$h_0(x) > g_0(x) \qquad \text{for some } x \in U. \tag{7}$$

Then there exist $r > 0$ and $h \in H$ such that if $g \in H$ and

$$\{x \in X: g(x) \geqslant h(x)\} \qquad \text{is a neighborhood of } E, \tag{8}$$

then

$$\| g - g_0 \| \geqslant r.$$

The companion result with the inequality signs in both (7) *and* (8) *reversed is also true.*

Proof. We consider first the case $g_0 = 0$. Assume that the statement of the lemma is false in this case. Suppose for a moment that we have constructed numbers $r_0, r_1,... > 0$ and functions $h_1, h_2,... \in H$ such that, for $k = 1, 2,...,$ if

$$g \in H \cap \operatorname{span}\{h_0,..., h_{k-1}\} \qquad \text{and} \qquad \{x \in X: g(x) \geqslant h_{k-1}(x)\} \tag{9}$$

is a neighborhood of E, then $\| g \| \geqslant r_{k-1}$,

$$\{x \in X: h_k(x) \geqslant h_{k-1}(x)\} \qquad \text{is a neighborhood of } E, \tag{10}$$

$$\| h_k \| < r_{k-1}. \tag{11}$$

Then, obviously, $h_k \in \operatorname{span}\{h_0,..., h_{k-1}\}$ for $k = 1, 2,...$ and this contradicts the fact that H is contained in the finite-dimensional subspace G of $C(X)$.

We now construct, inductively, the numbers $r_0, r_1,...$ and the functions $h_1, h_2,....$ Suppose that for some $\alpha_0 \in \mathbb{R}$, $\{x \in X: \alpha_0 h_0(x) \geqslant h_0(x)\}$ is a neighborhood of E. Then, since by hypothesis h_0 assumes a positive value in this neighborhood of E, $\alpha_0 h_0(x) \geqslant h_0(x) > 0$ for some $x \in X$. It follows that $\alpha_0 \geqslant 1$ and this implies that $\| \alpha_0 h_0 \| \geqslant \| h_0 \|$. Set $r_0 = \| h_0 \|$. Then (9) holds for $k = 1$ and, by the assumption that the lemma is false with $r = r_0$ and $h = h_0$, there exists $h_1 \in H$ such that (10) and (11) hold for $k = 1$.

Now suppose that, for some integer $n \geqslant 1$, $r_0,..., r_{n-1} > 0$ and $h_1,..., h_n \in H$ have been constructed such that (9)–(11) hold for $k = 1,..., n$. If, for some $\alpha_0,..., \alpha_n \in \mathbb{R}$, $\{x \in X: \alpha_0 h_0(x) + \cdots + \alpha_n h_n(x) \geqslant h_n(x)\}$ is a neighborhood of E, then, since by hypothesis h_0 assumes a positive value in the intersection of this neighborhood of E with the neighborhood $\{x \in X: h_n(x) \geqslant \cdots \geqslant h_1(x) \geqslant h_0(x)\}$ of E, $\alpha_0 h_0(x) + \cdots + \alpha_n h_n(x) \geqslant h_n(x) \geqslant \cdots \geqslant h_1(x) \geqslant h_0(x) > 0$ for some $x \in X$. For this x, $(|\alpha_0| + \cdots + |\alpha_n|) h_n(x) \geqslant |\alpha_0| h_0(x) + \cdots + |\alpha_n| h_n(x) \geqslant \alpha_0 h_0(x) + \cdots + \alpha_n h_n(x) \geqslant h_n(x) > 0$ and this implies that $|\alpha_0| + \cdots + |\alpha_n| \geqslant 1$. Since all norms on the $(n+1)$-dimensional subspace span$\{h_0,\ldots,h_n\}$ of G are equivalent, there exists $r_n > 0$ such that (9) holds for $k = n+1$. By the assumption that the lemma is false with $r = r_n$ and $h = h_n$, there exists $h_{n+1} \in H$ such that (10) and (11) hold for $k = n+1$. This establishes the lemma in case $g_0 = 0$.

For general g_0, the lemma follows from the special case applied to $H - g_0$, E and 0, $h_0 - g_0 \in H - g_0$. The result with reversed inequalities follows from the original version applied to $-H$, E and $-h_0$, $-g_0 \in -H$. The proof is complete. ∎

LEMMA 2. *Let $f \in C(X) \sim G$, let $h \in P(f)$ and let U be a neighborhood of $E(f - P(f))$. Set $H = \{g \in P(f)$: g coincides with h* in *$U\}$. Then for every $\varepsilon > 0$, there exists an $f_\varepsilon \in C(X)$ with the properties*

$$\|f_\varepsilon - f\| \leqslant \varepsilon, \tag{12}$$

$$d(f_\varepsilon) = d(f) \qquad \text{and} \qquad H \subset P(f_\varepsilon) \subset P(f), \tag{13}$$

if $h \in \tilde{H} \subset H$ and $\tilde{H}$ is convex, then

$$E^+(f - \tilde{H}) \subset E^+(f_\varepsilon - \tilde{H}) \subset E^+(f - \tilde{H}) \cup U$$

and

$$E^-(f - \tilde{H}) \subset E^-(f - \tilde{H}) \subset E^-(f - \tilde{H}) \cup U, \tag{14}$$

if $g \in P(f_\varepsilon)$, then $\{x \in X: g(x) \geqslant h(x)\}$ is a neighborhood of $E^+(f - P(f))$ and $\{x \in X: g(x \leqslant h(x)\}$ is a neighborhood of $E^-(f - P(f))$. (15)

Proof. We consider first the case $h = 0$. Suppose that $0 < \varepsilon < d(f)$. Since $0 \in P(f)$,

$$\begin{aligned} E^+(f - P(f)) &= \{x \in X: f(x) = d(f) \text{ and } g(x) = 0 \\ &\qquad \text{for all } g \in P(f)\}, \\ E^-(f - P(f)) &= \{x \in X: f(x) = -d(f) \text{ and } g(x) = 0 \\ &\qquad \text{for all } g \in P(f)\}, \end{aligned} \tag{16}$$

and hence there exist disjoint open neighborhoods $U^+ \subseteq U$ and $U^- \subseteq U$ of $E^+(f - P(f))$ and $E^-(f - P(f))$, respectively, such that

$$\begin{aligned} f(x) &\geqslant d(f) - \varepsilon \qquad &&\text{for all } x \in U^+, \\ f(x) &= -d(f) + \varepsilon \qquad &&\text{for all } x \in U^- \end{aligned} \tag{17}$$

and (note that $P(f)$ is a compact subset of $C(X)$)

$$|g(x)| \leqslant d(f) - \varepsilon \qquad \text{for all } g \in P(f) \text{ and all } x \in U^+ \cup U^-. \tag{18}$$

Let $V^+ \subset U^+$ and $V^- \subset U^-$ be closed neighborhoods of $E^+(f-P(f))$ and $E^-(f-P(f))$, respectively, and define two functions ϕ, ψ: $X \to \mathbb{R}$ as

$$\begin{aligned} \phi(x) &= d(f) && \text{if} \quad x \in V^+, \\ &= -d(f) && \text{if} \quad x \in U^-, \\ &= f(x) && \text{otherwise}, \\ \psi(x) &= d(f) && \text{if} \quad x \in U^+, \\ &= -d(f) && \text{if} \quad x \in V^-, \\ &= f(x) && \text{otherwise}. \end{aligned} \tag{19}$$

It is easily verified that ϕ is upper semi-continuous, ψ is lower semi-continuous and that $\phi(x) \leqslant \psi(x)$ for all $x \in X$. By the classical Tong–Katětov Interpolation Theorem (see, e.g., [4]), there exists an $f_\varepsilon \in C(X)$ such that

$$\phi(x) \leqslant f_\varepsilon(x) \leqslant \psi(x) \qquad \text{for all } x \in X, \tag{20}$$

and we claim that this f_ε has all the desired properties. By (19) and (20) we have

$$\begin{aligned} f_\varepsilon(x) &= f(x) && \text{for } x \in X \sim (U^+ \cup U^-), \\ f_\varepsilon(x) &= d(f) && \text{for } x \in V^+ \end{aligned}$$

and

$$\begin{aligned} f_\varepsilon(x) &= -d(f) && \text{for } x \in V^-, \\ f(x) \leqslant f_\varepsilon(x) &\leqslant d(f) && \text{for } x \in U^+ \sim V^+ \end{aligned}$$

and

$$-d(f) \leqslant f_\varepsilon(x) \leqslant f(x) \qquad \text{for } x \in U^- \sim V^-. \tag{21}$$

By (17) and (21), $\|f_\varepsilon - f\| \leqslant \varepsilon$, i.e., f_ε satisfies (12). By (21) (note that $d(f) = \|f\|$), $\|f_\varepsilon\| = \|f\|$, and by (16) and (21) (note that $E^+(f-P(f)) \subset V^+$ and $E^-(f-P(f)) \subset V^-$), $f_\varepsilon(x) = f(x)$ for all $x \in E(f-P(f))$. These two facts imply (see, e.g., [2]) that $d(f_\varepsilon) = d(f)$, and this together with (21) implies (note that $U^+ \cup U^- \subset U$) that $H \subset P(f_\varepsilon)$. To prove that also $P(f_\varepsilon) \subset P(f)$ is a little cumbersome. Let $g \in P(f_\varepsilon)$. Since $0 \in P(f_\varepsilon)$ and $P(f_\varepsilon)$ is convex, $\alpha g \in P(f_\varepsilon)$ for all $0 \leqslant \alpha \leqslant 1$, and therefore, by (17) and (21),

$$|f(x)-\alpha g(x)|=|f_\varepsilon(x)-\alpha g(x)|\leqslant d(f_\varepsilon)$$
$$\text{for all } x\in X\sim(U^+\cup U^-),\ 0\leqslant\alpha\leqslant 1,$$
$$f(x)-\alpha g(x)\leqslant f_\varepsilon(x)-\alpha g(x)\leqslant d(f_\varepsilon)$$
$$\text{for all } x\in U^+,\ 0\leqslant\alpha\leqslant 1,$$
$$f(x)-\alpha g(x)\geqslant f_\varepsilon(x)-\alpha g(x)\geqslant -d(f_\varepsilon)$$
$$\text{for all } x\in U^-,\ 0\leqslant\alpha\leqslant 1. \tag{22}$$

Since $\|g\|\leqslant\|f_\varepsilon-g\|+\|f_\varepsilon\|=2\|f_\varepsilon\|$, by (17),

$$f(x)-\tfrac{1}{2}g(x)\geqslant d(f)-\varepsilon-d(f_\varepsilon)\qquad\text{for all } x\in U^+,$$

and

$$f(x)-\tfrac{1}{2}g(x)\leqslant -d(f)+\varepsilon+d(f_\varepsilon)\qquad\text{for all } x\in U^-,$$

and these two inequalities combined with the inequalities (22) for $\alpha=\frac{1}{2}$ show (note that $d(f_\varepsilon)=d(f)$) that $\frac{1}{2}g\in P(f)$. Then, however, by (18), $|g(x)|\leqslant 2(d(f)-\varepsilon)$ for all $x\in U^+\cup U^-$ and therefore, again by (17),

$$f(x)-g(x)\geqslant d(f)-\varepsilon-2(d(f)-\varepsilon)=-d(f)+\varepsilon\qquad\text{for all } x\in U^+,$$

and

$$f(x)-g(x)\leqslant -d(f)+\varepsilon+2(d(f)-\varepsilon)=d(f)-\varepsilon\qquad\text{for all } x\in U^-.$$

These two inequalities combined with the inequalities (22) for $\alpha=1$ show that $g\in P(f)$. Thus $P(f_\varepsilon)\subset P(f)$, and this completes the proof that f_ε satisfies (13). Let $\tilde{H}$ be a convex subset of H which contains 0. Since $0\in\tilde{H}$,

$$E^+(f-\tilde{H})=\{x\in X: f(x)=d(f)\text{ and } g(x)=0\text{ for all } g\in\tilde{H}\},$$
$$E^-(f-\tilde{H})=\{x\in X: f(x)=-d(f)\text{ and } g(x)=0\text{ for all } g\in\tilde{H}\}$$

and (note that $\tilde{H}\subset P(f_\varepsilon)$ and $d(f_\varepsilon)=d(f)$)

$$E^+(f_\varepsilon-\tilde{H})=\{x\in X: f_\varepsilon(x)=d(f)\text{ and } g(x)=0\text{ for all } g\in\tilde{H}\},$$
$$E^-(f_\varepsilon-\tilde{H})=\{x\in X: f_\varepsilon(x)=-d(f)\text{ and } g(x)=0\text{ for all } g\in\tilde{H}\}.$$

Now by (21), if $x\in X$ and $|f(x)|=d(f)$, then $f_\varepsilon(x)=f(x)$. Thus $E^+(f-\tilde{H})\subset E^+(f_\varepsilon-\tilde{H})$, and $E^-(f-\tilde{H})\subset E^-(f_\varepsilon-\tilde{H})$. Again by (21) (note that $U^+\cup U^-\subset U$), if $x\in X$, $|f_\varepsilon(x)|=d(f)$ and $f(x)\neq f_\varepsilon(x)$, then $x\in U$. Thus $E^+(f_\varepsilon-\tilde{H})\subset E^+(f-\tilde{H})\cup U$ and $E^-(f_\varepsilon-\tilde{H})\subset E^-(f-\tilde{H})\cup U$. Thus f_ε satisfies (14). Once more by (21), if $g\in P(f_\varepsilon)$, then $g(x)\geqslant 0$ for all $x\in V^+$ and $g(x)\leqslant 0$ for all $x\in V^-$. Thus (note that V^+ is a neighborhood of $E^+(f-P(f))$ and V^- is a neighborhood of $E^-(f-P(f))$), f_ε satisfies (15) and we are done with the case $h=0$.

The general case of the lemma follows from the special case applied to $f-h$, $0 \in P(f-h)$ and U (note that $d(f-h)=d(f)$ and therefore $P(f-h)=P(f)-h$ and $E^\pm((f-h)-P(f-h))=E^\pm(f-P(f))$. The proof is complete. ∎

LEMMA 3. *Let S^* be a continuous selection for P, let P^* be the lower semi-continuous kernel of P induced by S^*, and let $f \in C(X)$ be such that $P^*(f) \neq P(f)$. Then for every $\varepsilon > 0$ there exists an $f_\varepsilon \in C(X)$ such that $\|f_\varepsilon - f\| \leqslant \varepsilon$ and $P(f_\varepsilon) = P^*(f)$.*

Proof. Let $P(f)=H_0 \supset H_1 \supset \cdots \supset \{S^*f\}$ be as in the Definition, and, for $k=1, 2,\ldots$, set

$$U_k = \text{interior of } \{x \in X\colon g(x) = S^*f(x) \text{ for all } g \in H_k\}.$$

Then, for each $k=1, 2,\ldots, U_k$ is an open neighborhood of $E(f-H_{k-1})$, $H_k = \{g \in H_{k-1}\colon g$ coincides with S^*f in $U_k\}$, and $U_k \subset U_{k+1}$. Suppose for a moment that for each $k=1, 2,\ldots$ and for every $\varepsilon > 0$ we have constructed a function $f_{k,\varepsilon} \in C(X)$ such that

$$\|f_{k,\varepsilon} - f\| \leqslant k\varepsilon, \tag{23}$$

$$d(f_{k,\varepsilon}) = d(f) \qquad \text{and} \qquad P(f_{k,\varepsilon}) = H_k, \tag{24}$$

if $S^*f \in \tilde{H} \subset H_k$ and $\tilde{H}$ is convex, then

$$E^+(f-\tilde{H}) \subset E^+(f_{k,\varepsilon} - \tilde{H}) \subset E^+(f-\tilde{H}) \cup U_k$$

and

$$E^-(f-\tilde{H}) \subset E^-(f_{k,\varepsilon} - \tilde{H}) \subset E^-(f-\tilde{H}) \cup U_k. \tag{25}$$

Let $k \geqslant 1$ be the smallest integer such that $P^*(f) = H_{k-1}$ (see the Definition). Since $P^*(f) \neq P(f)$ by hypothesis, we have that $k \geqslant 2$. By (23) and (24) the functions $f_\varepsilon = f_{k-1,\varepsilon/(k-1)}$, $\varepsilon > 0$ have the required properties.

We now construct the functions $f_{k,\varepsilon}$ inductively. First we prove

> if $g \in P(f)$ then $\{x \in X\colon g(x) \leqslant S^*f(x)\}$ is a neighborhood of $E^+(f-P(f))$ and $\{x \in X\colon g(x) \geqslant S^*f(x)\}$ is a neighborhood of $E^-(f-P(f))$. (26)

Assume that the first statement of (26) is false. Then $E^+(f-P(f))$ is non-empty and there exists an $h_0 \in P(f)$ with the property that for every neighborhood U of $E^+(f-P(f))$, $h_0(x) > S^*f(x)$ for some $x \in U$. By

Lemma 1 applied to $P(f)$, $E^+(f-P(f))$ and S^*f, h_0 there exist $r>0$ and $h\in P(f)$ such that if $g\in P(f)$ and $\{x\in X: g(x)\geqslant h(x)\}$ is a neighborhood of $E^+(f-P(f))$, then $\|g-S^*f\|\geqslant r$, and by Lemma 2 applied to f, h and X (note that our hypothesis $P^*(f)\neq P(f)$ implies that $f\notin G$), for every $\varepsilon>0$ there exists an $\tilde{f}_{1,\varepsilon}\in C(X)$ such that

$$\|\tilde{f}_{1,\varepsilon}-f\|\leqslant\varepsilon, \qquad P(\tilde{f}_{1,\varepsilon})\subset P(f) \qquad \text{and if} \qquad g\in P(\tilde{f}_{1,\varepsilon})$$

then

$$\{x\in X: g(x)\geqslant h(x)\} \qquad \text{is a neighborhood of} \quad E^+(f-P(f)).$$

It follows that $\tilde{f}_{1,\varepsilon}\to f$ as $\varepsilon\to 0$ and that $\|g-S^*f\|\geqslant r$ for every $g\in P(\tilde{f}_{1,\varepsilon})$ and every $\varepsilon>0$. This contradicts the fact that S^* is a continuous selection for P. Thus the first statement of (26) is true. The same argument with $E^-(f-P(f))$ in the place of $E^+(f-P(f))$ and the inequalities reversed, where applicable, shows that the second statement of (26) is also true. Next we observe that by Lemma 2 (applied to f, S^*f and U_1), for every $\varepsilon>0$ there exists an $f_{1,\varepsilon}\in C(X)$ such that

$$\|f_{1,\varepsilon}-f\|\leqslant\varepsilon,$$

$$d(f_{1,\varepsilon})=d(f) \qquad \text{and} \qquad H_1\subset P(f_{1,\varepsilon})\subset P(f),$$

if $S^*f\in\tilde{H}\subset H_1$ and $\tilde{H}$ is convex, then $E^+(f-\tilde{H})\subset E^+(f_{1,\varepsilon}-\tilde{H})\subset E^+(f-\tilde{H})\cup U_1$ and $E^-(f-\tilde{H})\subset E^-(f_{1,\varepsilon}-\tilde{H})\subset E^-(f-\tilde{H})\cup U_1$, and

if $g\in P(f_{1,\varepsilon})$ then $\{x\in X: g(x)\geqslant S^*f(x)\}$ is a neighborhood of $E^+(f-P(f))$ and $\{x\in X: g(x)\leqslant S^*f(x)\}$ is a neighborhood of $E^-(f-P(f))$. (27)

Let $\varepsilon>0$. It follows from (26) and (27) that every $g\in P(f_{1,\varepsilon})$ coincides with S^*f in some neighborhood of $E(f-P(f))$. Thus $P(f_{1,\varepsilon})\subset H_1$. Since by (27) also $H_1\subset P(f_{1,\varepsilon})$, we have that $P(f_{1,\varepsilon})=H_1$ and this together with (27) shows that the function $f_{1,\varepsilon}$ satisfies (23)–(25) for $k=1$.

Now suppose that for some integer $n\geqslant 1$ and for every $\varepsilon>0$ we have constructed a function $f_{n,\varepsilon}\in C(X)$ such that (23)–(25) hold for $k=n$. First we prove

if $g\in H_n$ then $\{x\in X: g(x)\leqslant S^*f(x)\}$ is a neighborhood of $E^+(f-H_n)$ and $\{x\in X: g(x)\leqslant S^*f(x)\}$ is a neighborhood of $E^-(f-H_n)$. (28)

Assume that the first statement of (28) is false. Then $E^+(f-H_n)$ is nonempty and there exists an $h_0\in H_n$ with the property that for every neighborhood U of $E^+(f-H_n)$, $h_0(x) > S^*f(x)$ for some $x \in U$. By Lemma 1 (applied to H_n, $E^+(f-H_n)$ and S^*f, h_0) there exist $r > 0$ and $h \in H_n$ such that

> if $g \in H_n$ and $\{x \in X: g(x) \geqslant h(x)\}$ is a neighborhood of $E^+(f-H_n)$, then $\|g - S^*f\| \geqslant r$,

and for every $\varepsilon > 0$, by Lemma 2 applied to $f_{n,\varepsilon}$, h and X (note that by (24) for $k = n, f_{n,\varepsilon} \notin G$ and $h \in P(f_{n,\varepsilon})$), there exists an $\tilde{f}_{n+1,\varepsilon} \in C(X)$ such that

> $\|\tilde{f}_{n+1,\varepsilon} - f_{n,\varepsilon}\| \leqslant \varepsilon$, $P(\tilde{f}_{n+1,\varepsilon}) \subset P(f_{n,\varepsilon})$ and if $g \in P(\tilde{f}_{n+1,\varepsilon})$ then $\{x \in X: g(x) \geqslant h(x)\}$ is a neighborhood of $E^+(f_{n,\varepsilon} - P(f_{n,\varepsilon}))$.

It follows (note (23) for $k = n$) that $\|\tilde{f}_{n+1,\varepsilon} - f\| \leqslant \|\tilde{f}_{n+1,\varepsilon} - f_{n,\varepsilon}\| + \|f_{n,\varepsilon} - f\| \leqslant (n+1)\,\varepsilon$ for every $\varepsilon > 0$ and (note that by (24) and (25) for $k = n$, $E^+(f-H_n) \subset E^+(f_{n,\varepsilon} - P(f_{n,\varepsilon})))$ that $\|g - S^*f\| \geqslant r$ for every $g \in P(\tilde{f}_{n+1,\varepsilon})$ and for every $\varepsilon > 0$. This contradicts the fact that S^* is a continuous selection for P. Thus the first statement of (28) is true. The same argument with $E^-(f-H_n)$ and $E^-(f_{n,\varepsilon} - P(f_{n,\varepsilon}))$ in the place of $E^+(f-H_n)$ and $E^+(f_{n,\varepsilon} - P(f_{n,\varepsilon}))$, respectively, and with the inequalities reversed where applicable, shows that the second statement of (28) is also true. Next we observe that for every $\varepsilon > 0$, by Lemma 2 (applied to $f_{n,\varepsilon}$, S^*f and U_{n+1} and noting that by (24) and (25) for $k = n$, $E(f_{n,\varepsilon} - P(f_{n,\varepsilon})) \subset E(f - H_n) \cup U_n$, and that U_{n+1} is a neighborhood of $E(f-H_n) \cup U_n$) there exists an $f_{n+1,\varepsilon} \in C(X)$ such that

$$\|f_{n+1,\varepsilon} - f_{n,\varepsilon}\| \leqslant \varepsilon, \tag{29}$$

$$d(f_{n+1,\varepsilon}) = d(f_{n,\varepsilon}) \qquad \text{and} \qquad H_{n+1} \subset P(f_{n+1,\varepsilon}) \subset P(f_{n,\varepsilon}), \tag{30}$$

if $S^*f \in \tilde{H} \subset H_{n+1}$ and $\tilde{H}$ is convex, then

$$E^+(f_{n,\varepsilon} - \tilde{H}) \subset E^+(f_{n+1,\varepsilon} - \tilde{H}) \subset E^+(f_{n,\varepsilon} - \tilde{H}) \cup U_{n+1}$$

and

$$E^-(f_{n,\varepsilon} - \tilde{H}) \subset E^-(f_{n+1,\varepsilon} - \tilde{H}) \subset E^-(f_{n,\varepsilon} - \tilde{H}) \cup U_{n+1}, \tag{31}$$

if $g \in P(f_{n+1,\varepsilon})$ then $\{x \in X: g(x) \geqslant S^*f(x)\}$ is a neighborhood of $E^+(f_{n,\varepsilon} - P(f_{n,\varepsilon}))$ and $\{x \in X: g(x) \leqslant S^*f(x)\}$ is a neighborhood of $E^-(f_{n,\varepsilon} - P(f_{n,\varepsilon}))$. (32)

Let $\varepsilon > 0$. By (23) for $k = n$ and (29), $\|f_{n+1,\varepsilon} - f\| \leqslant \|f_{n+1,\varepsilon} - f_{n,\varepsilon}\| + \|f_{n,\varepsilon} - f\| \leqslant (n+1)\,\varepsilon$. Thus $f_{n+1,\varepsilon}$ satisfies (23) for $k = n+1$. By (24) for $k = n$ and (29), $d(f_{n+1,\varepsilon}) = d(f_{n,\varepsilon}) = d(f)$. Since by (24) for $k = n$ and (30), $P(f_{n+1,\varepsilon}) \subset P(f_{n,\varepsilon}) = H_n$, and by (24) and (25) for $k = n$, $E^+(f-H_n) \subset$

$E^+(f_{n,\varepsilon} - P(f_{n,\varepsilon}))$ and $E^-(f - H_n) \subset E^-(f_{n,\varepsilon} - P(f_{n,\varepsilon}))$, it follows from (28) and (32) that every $g \in P(f_{n+1,\varepsilon})$ coincides with S^*f in some neighborhood of $E(f - H_n)$. This implies that $P(f_{n+1,\varepsilon}) \subset H_{n+1}$. Since by (30) also $H_{n+1} \subset P(f_{n+1,\varepsilon})$, we have that $P(f_{n+1,\varepsilon}) = H_{n+1}$. Thus $f_{n+1,\varepsilon}$ satisfies (24) for $k = n + 1$. Since $H_n \supset H_{n+1}$ and $U_n \subset U_{n+1}$, it follows from (25) for $k = n$ and (31) that $f_{n+1,\varepsilon}$ satisfies (25) for $k = n + 1$ and we are done. ∎

THEOREM. *If S^* is a continuous selection for P and if P^* is the lower semi-continuous kernel of P induced by S^*, then for every $f \in C(X)$*

$$P^*(f) = \bigcup \{Sf\colon S \textit{ a continuous selection for } P\}.$$

Proof. For every $f \in C(X)$, set

$$Q(f) = \bigcup \{Sf\colon S \text{ a continuous selection for } P\}.$$

We show first that $Q(f) \subset P^*(f)$ for every $f \in C(X)$. Let $f \in C(X)$. If $P^*(f) = P(f)$, then $Q(f) \subset P^*(f)$ by the definition of Q. We suppose now that $P^*(f) \neq P(f)$. By Lemma 3 there exist $f_1, f_2, \ldots \in C(X)$ such that $f_n \to f$ as $n \to \infty$ and $P(f_n) = P^*(f)$ for all n. Consequently, for every continuous selection S for P, $Sf_n \to Sf$ as $n \to \infty$ and $Sf_n \in P^*(f)$ for all n, and therefore, since $P^*(f)$ is closed, $Sf \in P^*(f)$. Thus $Q(f) \subset P^*(f)$ also in this case.

It remains to show that $P^*(f) \subset Q(f)$ for every $f \in C(X)$. By the consequence of results of Michael stated in (2), it suffices to prove that P^* is lower semi-continuous. Let $f \in C(X)$, let $g \in P^*(f)$, and let $f_1, f_2, \ldots \in C(X)$ be such that $f_n \to f$ as $n \to \infty$. We must prove that there exist $g_n \in P^*(f_n)$, $n = 1, 2, \ldots$, such that $g_n \to g$ as $n \to \infty$. If $f \in G$, then obviously $g = f$, and it is easily verified (and well known) that $g_n \to f$ as $n \to \infty$ no matter how the g_n are chosen in ($P(f_n)$ and hence in) $P^*(f_n)$. Suppose now that $f \notin G$. Since the relative interior of $P^*(f)$ is dense in $P^*(f)$, we may restrict attention to the case that g belongs to the relative interior of $P^*(f)$. One easily verifies that in this case $E(f - P^*(f)) = E(f - g)$. Since $U = \{x \in X\colon g(x) = S^*f(x)\}$ is a neighborhood of $E(f - P^*(f))$, it follows that there exists $0 < \varepsilon < d(f)$ such that

$$|f(x) - g(x)| \leqslant d(f) - \varepsilon \qquad \text{for all } x \in X \sim U.$$

Now, for $n = 1, 2, \ldots$, choose $\tilde{f}_n \in C(X)$ such that $\|\tilde{f}_n - f\| \leqslant 1/n$ and $P(\tilde{f}_n) = P^*(f_n)$ (if $P^*(f_n) = P(f_n)$ set $\tilde{f}_n = f_n$ and if $P^*(f_n) \neq P(f_n)$ appeal to Lemma 3) and set $\tilde{g}_n = g + S^*\tilde{f}_n - S^*f$. Then

$$|\tilde{f}_n(x) - \tilde{g}_n(x)| = |\tilde{f}_n(x) - S^*\tilde{f}_n(x)| \leqslant d(\tilde{f}_n)$$

$$\text{for all } x \in U \text{ and all } n,$$

and

$$|\tilde{f}_n(x) - \tilde{g}_n(x)| \leqslant |\tilde{f}_n(x) - f(x)| + |f(x) - g(x)| + |S^*\tilde{f}_n(x) - S^*f(x)|$$
$$\leqslant \|\tilde{f}_n - f\| + (d(f) - \varepsilon) + \|S^*\tilde{f}_n - S^*f\|$$
$$\text{for all } x \in X \sim U \text{ and all } n.$$

Observing that $\tilde{f}_n \to f$ as $n \to \infty$ and therefore $S^*\tilde{f}_n \to S^*\tilde{f}$ and $d(\tilde{f}_n) \to d(f)$ as $n \to \infty$, we infer from the last inequality that

$$|\tilde{f}_n(x) - \tilde{g}_n(x)| \leqslant d(\tilde{f}_n) \qquad \text{for all } x \in X \sim U \text{ and all sufficiently large } n,$$

and combining this with the next-to-last inequality above shows that $\tilde{g}_n \in P(\tilde{f}_n)$ for all sufficiently large n. Since $P(\tilde{f}_n) = P^*(f_n)$ for all n and $\tilde{g}_n \to g$ as $n \to \infty$, we are done. ∎

It is an immediate consequence of our Theorem that P^* is entirely independent of the particular choice of S^*. Thus, given that P admits a continuous selection, we can speak of *the lower semi-continuous kernel P^* of P.*

3. 1-Dimensional Subspaces of $C(X)$

For this section let $G = \text{span}\{g\}$ for some non-zero $g \in C(X)$, and set $Z = \{x \in X: g(x) = 0\}$. Lazar *et al.* [6] showed that P admits a continuous selection in precisely the following four mutually exclusive cases.

Case 1. Z is empty.

Case 2. The interior of Z is empty, the boundary of Z is a singleton and one of $\{x \in X: g(x) \geqslant 0\}$ and $\{x \in X: g(x) \leqslant 0\}$ is a neighborhood of Z.

Case 3. Z is non-empty and open.

Case 4. The interior of Z is non-empty, the boundary of Z is a singleton and one of $\{x \in X: g(x) \geqslant 0\}$ and $\{x \in X: g(x) \leqslant 0\}$ is a neighborhood of Z.

We show now that P possesses a unique continuous selection in precisely the first two cases.

In the first case, G is a Tchebycheff subspace of $C(X)$ and therefore $P(f)$ is a singleton for every $f \in C(X)$. Thus $P^*(f)$ is a singleton for every $f \in C(X)$.

In the second case, assume that for some $f \in C(X)$, g_1 and g_2 are distinct elements of $P^*(f)$. Then the set $\{x \in X: g_1(x) = g_2(x)\}$ on one hand is contained in Z and on the other hand is a neighborhood of $E(f-P(f))$ and thus has non-empty interior, a contradiction. Thus $P^*(f)$ is a singleton for every $f \in C(X)$.

In the third case, there exists an $f \in C(X)$ such that $P(f)$ is not a singleton. Then, since all elements of $P(f)$ coincide on $E(f-P(f))$, $E(f-P(f))$ is contained in Z. Since Z is open, it follows that $P^*(f)=P(f)$. Thus $P^*(f)$ is not a singleton.

In the fourth case, there exists an $f \in C(X)$ of norm 1 which vanishes off Z. One easily verifies that $P(f)=\{\alpha g\colon |\alpha| \leqslant 1\}$ and that Z is a neighborhood of $E(f-P(f))$. It follows that $P^*(f)=P(f)$. Thus $P^*(f)$ is not a singleton.

We note that the function constructed in the fourth case could have been constructed also in the third case. We chose not to do so because our argument in the third case actually shows that $P^*(f)=P(f)$ for all $f \in C(X)$; i.e., $P^*=P$, while (see Remark 4) $P^* \neq P$ in the fourth case. We treated the first two cases separately for the same reason.

4. Finite-Dimensional Subspaces of $C(X)$, X an Interval

For this section, let X be a non-degenerate closed interval $[a, b]$ of the real line. Nürnberger and Sommer (see [9]) showed that the first of the following two conditions implies the second.

Condition 1. G is a Weak–Tchebycheff subspace of $C(X)$ and no non-zero element of G has more than $\dim(G)$ distinct zeros.

Condition 2. P possesses a unique continuous selection.

We show now that these two conditions are actually equivalent. Suppose P admits a continuous selection but G does not satisfy the first condition. We must prove that P possesses more than one continuous selection. By results of Nürnberger and Sommer (see [9]) G is a Weak–Tchebycheff subspace of $C(X)$, some norm 1 function g in G vanishes on some non-degenerate subinterval of X, and there exist an integer $k \geqslant 1$ and points $a=x_0<x_1<\cdots<x_{k+1}=b$ such that, for $0 \leqslant i \leqslant k$, a function in G which vanishes on a non-degenerate subinterval of $[x_i, x_{i+1}]$, vanishes on $[x_i, x_{i+1}]$. It follows that g vanishes on $[x_j, x_{j+1}]$ for some $0 \leqslant j \leqslant k$. Now we distinguish two cases.

Suppose first that all functions in G vanish on $[x_j, x_{j+1}]$. Let $f \in C(X)$ be a norm 1 function which vanishes off $[x_j, x_{j+1}]$. Then $\{\alpha g\colon |\alpha| \leqslant 1\} \subset P(f)$ and $[x_j, x_{j+1}]$ is a neighborhood of $E(f-P(f))$. It follows that $P^*(f)=P(f)$. Thus $P^*(f)$ is not a singleton. The example shown in Fig. 1 of a 1-dimensional subspace G of a full spline space is typical for this case (note that this is Case 4 of Section 3).

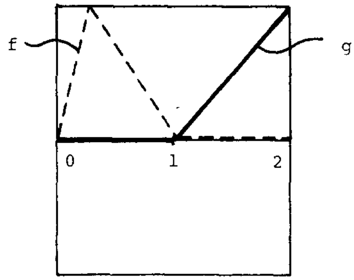

FIG. 1. $P^*(f)$ is not a singleton.

Suppose next that some function in G is non-zero somewhere in $[x_j, x_{j+1}]$ and therefore throughout some non-degenerate subinterval $[c, d]$ of $[x_j, x_{j+1}]$. Let $f \in C(X)$ be a norm 1 function which vanishes off $[c, d]$ and which alternates p times, i.e., there exist points $c < t_0 < t_1 < \cdots < t_p < d$ such that $f(t_i) = (-1)^i$ for $0 \leqslant i \leqslant p$, where p is sufficiently large. Then $d(f) = 1$, and by well-known zero properties of Weak–Tchebycheff spaces (see, e.g., [8, Theorem 2.45]), all elements of $P(f)$ vanish on $[c, d]$. Thus $\{\alpha g: |\alpha| \leqslant 1\} \subset P(f)$ and $[c, d]$ is a neighborhood of $E(f - P(f))$. It follows that $P^*(f) = P(f)$. Thus $P^*(f)$ is not a singleton. The typical example for this case is the space $G = \mathscr{S}_m(\Delta)$ of polynomial splines of order m with (simple) knots $\Delta = \{x_1, ..., x_k\}$ where $k \leqslant m$.

5. REMARKS

(1) The first application of our Theorem was made in [3], where we showed that for spline approximation continuous selections are non-unique whenever they exist. In [3] we also showed that for spline approximation the lower semi-continuous kernel P^* of P has an even nicer description than in general and we gave examples illustrating the definition of the sets $P^*(f)$.

(2) Brown [5] proved that if G has the property that none of its non-zero elements vanish on a non-empty open subset of X, then if there exists a continuous selection for P it is unique. This is an immediate consequence of our Theorem and the fact that for every $f \in C(X)$ any two elements of $P^*(f)$ coincide in a neighborhood of $E(f - P^*(f))$ (see our argument in Case 2 of Section 3).

(3) As we mentioned at the beginning of Section 2, our construction of the lower semi-continuous kernel P^* of P was inspired by the unpublished pointwise criterion for lower semi-continuity of P quoted in (3). It is not too much of a surprise therefore, that in case P admits a continuous selection,

this criterion follows immediately from our results on P^*: Note first that $P^*(f) = P(f)$ iff all elements of $P(f)$ coincide in a neighborhood of $E(f - P(f))$. Now suppose that P is lower semi-continuous at f. If $g \in P(f)$, by Lemma 3 there exist $f_n \to f$ such that $P(f_n) = P^*(f)$, and since P is lower semi-continuous at f, there exist $g_n \in P(f_n)$ such that $g_n \to g$. Thus $P^*(f) = P(f)$. Conversely, suppose that $P^*(f) = P(f)$. If $g \in P(f)$ and $f_n \to f$, then since P^* is lower semi-continuous, there exist $g_n \in P^*(f_n)$ such that $g_n \to g$. Thus P is lower semi-continuous at f.

(4) In concluding this paper, we feel obliged to say a word or two about the fact that the pointwise criterion for lower semi-continuity of P remained unpublished. In [2], Blatter *et al.* proved that P is lower semi-continuous iff for every $f \in C(X)$ such that $0 \in P(f)$ the set $Z(P(f)) = \{x \in X: g(x) = 0$ for all $g \in P(f)\}$ is open; and this global criterion for lower semi-continuity of P is, of course, a modification of the pointwise criterion above. The modification consists in passing, for $f \in C(X)$ such that $0 \in P(f)$, from the set $E(f - P(f))$ to the larger—and often strictly larger—set $Z(P(f))$. This passage facilitates certain conclusions, e.g., that for connected X, P is lower semi-continuous iff G is Tchebycheff, but results in a loss of information without which the present paper would have been impossible.

References

1. J. Blatter, "Zur Stetigkeit von mengenwertigen metrischen Projektionen," IIM–Bericht 39/67, Univ. Bonn, 1967.
2. J. Blatter, P. D. Morris, and D. E. Wulbert, Continuity of the set-valued metric projection. *Math. Ann.* **178** (1968), 12–24.
3. J. Blatter and L. Schumaker, Continuous selections and maximal alternators for spline approximation, *J. Approx. Theory*, to appear.
4. J. Blatter and G. L. Seever, Interposition and lattice cones of functions. *Trans. Amer. Math. Soc.* **222** (1967), 65–96.
5. A. L. Brown, On continuous selections for metric projections in spaces of continuous functions. *J. Funct. Anal.* **8** (1971), 431–449.
6. A. J. Lazar, D. E. Wulbert, and P. D. Morris, Continuous selections for metric projections. *J. Funct. Anal.* **3** (1969), 193–216.
7. E. Michael, Continuous selections, I. *Ann. Math.* **63** (1956), 361–382.
8. L. L. Schumaker, "Spline Functions: Basic Theory," Wiley–Interscience, New York, 1981.
9. M. Sommer, Characterization of continuous selections of the metric projection for a class of weak Chebyshev spaces. *SIAM J. Math. Anal.* **13** (1982), 280–294.